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Systematics of flux tubes in the dual Ginzburg–Landau theory and Casimir scaling hypothesis: folklore and lattice facts

Y. Koma[1], M. Koma (Takayama)[2,*]

[1] Institute for Theoretical Physics, Kanazawa University, Kanazawa, Ishikawa 920-1192, Japan
[2] Research Center for Nuclear Physics (RCNP), Osaka University, Mihogaoka 10-1, Ibaraki, Osaka 567-0047, Japan

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Abstract. The ratios between the string tensions σ_D of color-electric flux tubes in higher and fundamental SU(3) representations, $d_D \equiv \sigma_D/\sigma_{\mathbf{3}}$, are systematically studied in a Weyl symmetric formulation of the DGL theory. The ratio is found to depend on the Ginzburg–Landau (GL) parameter, $\kappa \equiv m_\chi/m_B$, the mass ratio between the monopoles (m_χ) and the masses of the dual gauge bosons (m_B). While the ratios d_D follow a simple flux counting rule in the Bogomol'nyi limit, $\kappa = 1.0$, systematic deviations appear with increasing κ due to interactions between the fundamental flux inside a higher representation flux tube. We find that in a type-II dual superconducting vacuum near $\kappa = 3.0$ this leads to a consistent description of the ratios d_D as observed in lattice QCD simulations.

1 Introduction

The study of the static potential in QCD between color charges in various representations of the color group SU(3) is expected to discriminate between possible candidate confinement scenarios, i.e. between the sorts of non-perturbative vacuum proposed [1,2]. Recently, such static potentials have been investigated within SU(3) pure lattice gauge theory extracted from Wilson loops in various representations D [3,4]. In [3] the static potentials have been parameterized as a superposition of Coulomb, linear and constant terms, where the ratios among the string tensions governing the linear terms were found to be roughly equal to the ratios between the eigenvalues of the quadratic Casimir operator $C^{(2)}(D) = \langle D| \sum_{a=1}^{8} T^a T^a |D\rangle$ in the respective representation. This has been considered as a confirmation of the Casimir scaling hypothesis which is under discussion since long [1,2,5]. In [4], instead, the ratios of the static potentials themselves are analyzed, which show very good agreement with the Casimir ratios uniformly up to distances of $r \sim 1\,\mathrm{fm}$. Following these results, strong arguments have been raised stressing that QCD-vacuum models should reproduce Casimir scaling [6,7].

Recently, in [8] one of the present authors has examined the dual Ginzburg–Landau (DGL) theory [9–11] as a QCD-vacuum model under this aspect. In the DGL theory, the quark confinement mechanism is described by the formation of a color-electric flux tube due to a dual Meissner effect. The ratios between string tensions of charges in higher and fundamental SU(3) representations, $d_D \equiv \sigma_D/\sigma_{\mathbf{3}}$, have been calculated within the DGL theory cast into a Weyl symmetric formulation [12,13]. It has been shown that the ratios depend on the GL parameter, $\kappa \equiv m_\chi/m_B$. If one wants to reproduce the *exact* Casimir ratios d_D, one has to choose the GL parameters depending on the representations thus: $\kappa \sim 5.0$ for $D = \mathbf{8}$, $\mathbf{6}$, $\kappa \sim 7.0$ for $D = \mathbf{10}$, $\mathbf{15a}$, and $\kappa \sim 9.0$ for $D = \mathbf{27}$, $\mathbf{24}$, and $\mathbf{15s}$. There was no *unique* GL parameter which would have reproduced *all* Casimir ratios at once.

Does this result rule out the DGL theory as a QCD-vacuum model? Before hastily drawing thse conclusions, we would like to reconsider the lattice data. We find that the string tensions shown in [3] do *not obey exact* Casimir scaling. They have huge errors for higher representations. In [4] a clear signal of Casimir scaling for *static potentials* is obtained. However, it is not clear that this result carries over to string tensions. This depends on how one separates the string tension from the potential, since the tail of the short range Coulomb potential contributes to the apparent slope of the long range part of the potential. Hence, there is no reason that the ratio between string tensions is identical with that between potentials. On the theoretical side, it is not obvious that such group theoretical scaling appears for the string tension at distances where non-perturbative effects start to become important.

If the behavior of the ratio would be governed exclusively by the group theoretical factor, it would be natural to expect that Casimir scaling is manifest in *arbitrary* SU(N) gluodynamics. However, very recent studies of k-strings in SU(4) and SU(6) lattice gauge theories did not provide support for Casimir scaling but instead favored a sine-formula scaling [14,15].

* The family name has changed in May 2001 due to marriage

In the present paper we study the *systematics* of the ratios of string tensions among various representations, restricted to SU(3) gluodynamics, in the DGL theory following [8]. We want to see whether a certain unique value of the GL parameter can provide all the ratios of string tensions consistent with lattice data themselves, without any bias toward Casimir scaling. Using the method of [8] we can compare the DGL theory with the lattice data of [3], because only there string tensions have been extracted. Finally, we would like to speculate on which features of non-perturbative dynamics could be the origin of the representation dependence of the string tensions observed in lattice simulations.

2 The DGL theory

We briefly review how to calculate the string tensions in the DGL theory [8]. In order to treat the charges in various SU(3) representations systematically, we start from the Weyl symmetric form of the DGL theory [12,13]:

$$\mathcal{L}_{\rm DGL} = \sum_{i=1}^{3}\left[-\frac{1}{6g^2}\left((\partial\wedge B_i)_{\mu\nu} + 2\pi\sum_{j=1}^{3} m_{ij}\Sigma^{(e)}_{j\;\mu\nu}\right)^2 + \left|(\partial_\mu + \mathrm{i}B_{i\mu})\chi_i\right|^2 - \lambda\left(|\chi_i|^2 - v^2\right)^2\right], \qquad (1)$$

where $B_{i\mu}$ $(i=1,2,3)$, and χ_i $(i=1,2,3)$ denote the dual gauge field and the complex scalar monopole field, respectively. The dual gauge fields within the Weyl symmetric expression are subject to the constraint $\sum_{i=1}^{3} B_{i\mu} = 0$. A distinctive feature of the DGL Lagrangian is that the quark current $j^{(e)}_{j\;\mu}$ $(j=1,2,3)$ is represented as the boundary of a non-local color-electric Dirac string term $\Sigma^{(e)}_{j\;\mu\nu}$, as $j^{(e)}_{j\;\mu} = \partial^{\nu\,*}\Sigma^{(e)}_{j\;\mu\nu}$, which corresponds to the modified dual Bianchi identity. Note that $(\partial\wedge B_i)_{\mu\nu} \equiv \partial_\mu B_{i\nu} - \partial_\nu B_{i\mu}$ satisfies $\partial^{\nu\,*}(\partial\wedge B_i)_{\mu\nu} = 0$.

In this framework, the color-electric charge of the quark is specified by using the weight vector of the SU(3) algebra, $\boldsymbol{w}_j$ $(j=1,2,3)$, as $\boldsymbol{Q}^{(e)}_j \equiv e\boldsymbol{w}_j$, where $\boldsymbol{w}_1 = (1/2, 3^{1/2}/6)$, $\boldsymbol{w}_2 = \left(-1/2, 3^{1/2}/6\right)$, and $\boldsymbol{w}_3 = \left(0, -1/3^{1/2}\right)$. On the other hand, the color-magnetic charges of the monopole fields χ_i are expressed by the root vectors of the SU(3) algebra, $\boldsymbol{\epsilon}_i$, as $Q^{(m)}_i \equiv g\boldsymbol{\epsilon}_i$ $(i=1,2,3)$, where $\boldsymbol{\epsilon}_1 = \left(-1/2, 3^{1/2}/2\right)$, $\boldsymbol{\epsilon}_2 = \left(-1/2, -3^{1/2}/2\right)$, and $\boldsymbol{\epsilon}_3 = (1,0)$. The appearance of the matrix m_{ij} in the dual field strength tensor is due to the extended Dirac quantization condition between the color-electric and the color-magnetic charges, $\boldsymbol{Q}^{(m)}_i \cdot \boldsymbol{Q}^{(e)}_j = 2\pi m_{ij}$, where we have required $eg = 4\pi$. The entries of the matrix m_{ij} are integers expressed by means of the third-rank antisymmetric tensor ϵ_{ijk} as $m_{ij} = 2\boldsymbol{\epsilon}_i\cdot\boldsymbol{w}_j = \sum_{k=1}^{3}\epsilon_{ijk}$. Using the matrix m_{ij}, the dual gauge field is decomposed into two parts, $B_{i\mu} = B^{\rm reg}_{i\mu} + \sum_{j=1}^{3} m_{ij} B^{\rm sing}_{j\;\mu}$, where the singular part $B^{\rm sing}_{j\;\mu}$ is determined by the relation

$$(\partial\wedge B^{\rm sing}_j)_{\mu\nu} + 2\pi\Sigma^{(e)}_{j\;\mu\nu} = 2\pi C^{(e)}_{j\;\mu\nu} \quad (j=1,2,3), \qquad (2)$$

where $C^{(e)}_{j\;\mu\nu}$ is a Coulombic field which does not contain any Dirac string, given by[1]

$$C^{(e)}_{j\;\mu\nu}(x) = \frac{1}{4\pi^2}\int \mathrm{d}^4y\,\frac{1}{|x-y|^2}\,{}^*(\partial\wedge j^{(e)}_j(y))_{\mu\nu}. \qquad (3)$$

The two mass scales of the DGL theory are the mass of the dual gauge field $m_B = 3^{1/2}gv$ and the mass of the monopole field $m_\chi = 2(\lambda^{1/2})v$. The corresponding inverse masses are related to the thickness of the flux tube, given by the penetration depth of the color-electric field into the vacuum, and to the coherence length of the monopole field, respectively. In analogy to the usual superconductors, their ratio, $\kappa \equiv m_\chi/m_B$, is a label of the type of dual superconductivity of the vacuum.

3 Flux-tube solution

The flux-tube solution in the cylindrical symmetric system with translational invariance along the z axis is described, as a function of the two-dimensional radius r and the azimuthal angle φ, by the modulus of the monopole field $\phi_i(r) = |\chi_i(r)|$ and the regular part of the dual gauge field $B^{\rm reg}_i(r)\boldsymbol{e}_\varphi = [\tilde{B}^{\rm reg}_i(r)/r]\boldsymbol{e}_\varphi$. In this system, the contribution of the Coulomb term (3) can be set to be zero because the static charges are infinitely apart. Thus, (2) leads to $B^{\rm sing}_i(r)\boldsymbol{e}_\varphi = -(n^{(m)}_i/r)\boldsymbol{e}_\varphi$, with

$$n^{(m)}_i \equiv \sum_{j=1}^{3} m_{ij} n^{(e)}_j. \qquad (4)$$

Here $n^{(e)}_j$ is the winding number of the j-type color-electric Dirac string $\Sigma^{(e)}_{j\;\mu\nu}$, which can take various integers depending on the representation of the SU(3) color group to which the charges belong (see Table 1). The string tension of the flux tube is calculated as an energy per unit length in the z direction. We have

$$\sigma_D = 2\pi\sum_{i=1}^{3}\int_0^\infty r\mathrm{d}r\left[\frac{1}{3g^2}\left(\frac{1}{r}\frac{\mathrm{d}\tilde{B}^{\rm reg}_i}{\mathrm{d}r}\right)^2 + \left(\frac{\mathrm{d}\phi_i}{\mathrm{d}r}\right)^2 + \left(\frac{\tilde{B}^{\rm reg}_i - n^{(m)}_i}{r}\right)^2\phi_i^2 + \lambda(\phi_i^2 - v^2)^2\right]. \qquad (5)$$

In the Bogomol'nyi limit, $\kappa = m_\chi/m_B = 1.0$, one gets the saturated string tension analytically as [8]

$$\sigma_D = 2\pi v^2\sum_{i=1}^{3}\left|n^{(m)}_i\right| = 4\pi v^2(p+q). \qquad (6)$$

[1] The decomposition of the dual gauge field is also given in [16] in a more elegant way using differential forms

Table 1. Eigenvalues of the quadratic Casimir operators $C^{(2)}(D)$, and of $A^{(2)}(D)$, its restriction to the Cartan subgroup, for various SU(3) representations denoted by D with $[p,q]$ as the Dynkin index. $\{n_j^{(e)}\}$ classifies the winding number of the flux-tube solution in the DGL theory which belongs to the given SU(3) representation

D	$[p,q]$	$p+q$	$C^{(2)}(D)$	(ratio)	$A^{(2)}(D)$	(ratio)	$\{n^{(e)}_{j=1,2,3}\}$
3	[1,0]	1	4/3	–	1/3	–	$\{1,0,0\}$
8	[1,1]	2	3	9/4	1	3	$\{1,-1,0\}$
6	[2,0]		10/3	5/2	4/3	4	$\{2,0,0\}$
15a	[2,1]	3	16/3	4	7/3	7	$\{2,-1,0\}$
10	[3,0]		6	9/2	3	9	$\{3,0,0\}$
27	[2,2]	4	8	6	4	12	$\{2,-2,0\}$
24	[3,1]		23/3	23/4	13/3	13	$\{3,-1,0\}$
15s	[4,0]		28/3	7	16/3	16	$\{4,0,0\}$

Then the ratio of the string tension between a higher and the fundamental representation is simply given by

$$d_D = \sigma_D/\sigma_{\mathbf{3}} = p+q, \tag{7}$$

which is nothing but the sum $p+q$ of the Dynkin index of the representation D of the SU(3) group. In the general dual superconducting vacuum of type I ($\kappa < 1.0$) and of type II ($\kappa > 1.0$), one has to evaluate the whole expression (5) in its variational minimum by solving the field equations numerically.

4 Numerical result, new features and motivation

In Fig. 1, we show the ratios of the string tensions of the flux tubes, $d_D = \sigma_D/\sigma_{\mathbf{3}}$ for three values of the GL parameter, $\kappa = 1.0$, 3.0, and 9.0 (numerically obtained for $\kappa \neq 1.0$). We also plot the ratios of the string tensions obtained by the lattice simulations of [3]. To characterize two hypothetical cases under discussion, we plot also the ratios of eigenvalues of the quadratic Casimir operator evaluated in the highest weight state,

$$\begin{aligned} C^{(2)}(D) &= \langle D_{\max}|(T^3)^2 + (T^8)^2 + 2T^3|D_{\max}\rangle \\ &= \frac{1}{3}(p^2+pq+q^2) + (p+q), \end{aligned} \tag{8}$$

as well as its Abelian projected (Cartan restricted) values

$$\begin{aligned} A^{(2)}(D) &= \langle D_{\max}|(T^3)^2 + (T^8)^2|D_{\max}\rangle \\ &= \frac{1}{3}(p^2+pq+q^2). \end{aligned} \tag{9}$$

Here the relations $T^3|D_{\max}\rangle = (p+q)/(2)|D_{\max}\rangle$ and $T^8|D_{\max}\rangle = (p-q)/(2(3^{1/2}))|D_{\max}\rangle$ for the highest weight state $|D_{\max}\rangle$ have been used. Values for the lowest eight representations are tabulated in Table 1. We find

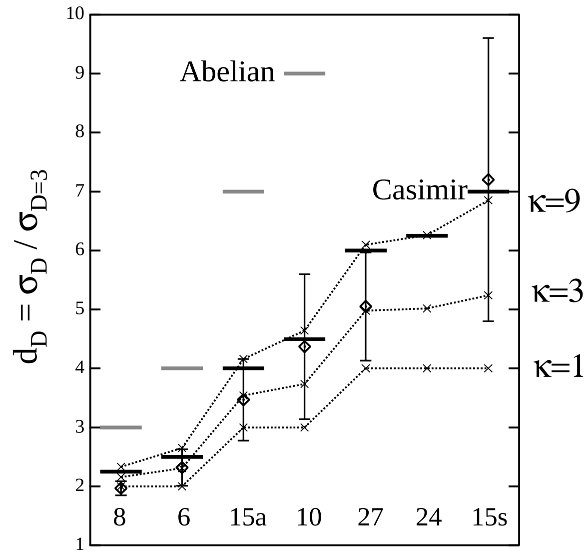

Fig. 1. The ratios of the string tensions of the flux tubes for various SU(3) representations, $d_D = \sigma_D/\sigma_{\mathbf{3}}$ for the GL parameters $\kappa = 1.0$, 3.0 and 9.0 (represented by crosses, each case connected by lines to guide the eye). The ratios of eigenvalues of the quadratic Casimir operators are shown as black bars. Restricted to the Cartan algebra (Abelian scaling), the ratios are shown as gray bars. For comparison the lattice data of [3] are also plotted (diamonds with error bars)

that the DGL result in the type-II dual superconducting vacuum near $\kappa = 3.0$ agrees well with all lattice data obtained in [3], albeit with big errors.

The mechanism of the κ dependence is understood as follows. In the Bogomol'nyi limit, $\kappa = 1.0$, the ratio between the string tensions of a higher and the fundamental representation satisfies the *flux counting rule*; the string tension σ_D is simply proportional to the number of the color-electric Dirac strings inside the flux tube, as seen from (6). With increasing κ, the interaction ranges of these fields get out of balance, and an excess of energy appears because of the interaction between fundamental flux tubes [17,18]. This leads to systematic deviations from the counting rule[2]. Note that the deviation of d_D from the counting rule grows toward higher representations D, since the number of fundamental fluxes which coexist in the flux tube of representation D increases as the sum $p+q$ of Dynkin indices.

On the other hand, we also find that the DGL result at $\kappa = 9.0$, for the deeply type-II vacuum, *uniformly* reproduces *Casimir-like* ratios by accident, through the deviations from the flux counting rule. This result does not contradict the previous one [8], where the GL parameters were searched which reproduce the *exact* Casimir ratio d_D for each D.

Since the DGL theory is constructed from QCD via the Abelian projection scheme, the objection has been

[2] In this analysis, the higher dimensional flux tube is assumed to be stable against splitting into fundamental ones. However there must be a certain minimal q–$\bar{q}$ distance, depending on the GL parameter. Otherwise this effect is not negligible

raised that one would then have *Abelian scaling* following $A^{(2)}(D)$ (9) instead of *Casimir scaling* following $C^{(2)}(D)$ (8) in the DGL theory. Abelian scaling, for instance, would give a ratio between the $D = \mathbf{8}$ and $D = \mathbf{3}$ representations as large as $A^{(2)}(\mathbf{8})/A^{(2)}(\mathbf{3}) = 3.0$, in clear distinction from the Casimir ratio 9/4 *and* the lattice value. We find that the ratios of the string tensions of flux tubes in the DGL theory are not steeply rising as dictated by Abelian scaling, although the Abelian projected theory has been claimed to have Abelian scaling realized not only at short distance but also in the long range force [6].

5 Summary

We have studied the string tensions of flux tubes associated with static charges in various SU(3) representations in the DGL theory, based on a manifestly Weyl symmetric procedure. We have found that a GL parameter near $\kappa = 3.0$ *can reproduce* the ratios of string tensions consistent with the lattice data [3]. We have also found that the ratios of string tensions are far from Abelian scaling at any finite value of κ. The DGL theory accidentally shows Casimir-like scaling for a deeply type-II vacuum with $\kappa = 9.0$. But there is no obvious relation to the eigenvalues of the Casimir operator.

The mechanism behind the systematic behavior of string tensions in the DGL theory can rather be understood as a result of the flux-tube dynamics. This includes the possibility that the lattice data contain a similar dynamical effect. We would like to emphasize that it is important to have more lattice results carefully interpreted, without bias toward Casimir scaling, before one is able to judge the viability of various QCD-vacuum models.

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